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Bayesian reliability-based robust design optimization of mechanical systems under both aleatory and epistemic uncertainties

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ABSTRACT

Uncertainties can be divided into two general categories: aleatory and epistemic. Conventional reliability-based robust design optimization approaches, which disregard epistemic uncertainties due to lack of knowledge about the physical nature of systems, have previously been developed. To overcome this weakness, unlike previous methods, a Bayesian reliability-based robust design optimization method is proposed in the presence of both aleatory and epistemic uncertainties. The proposed formulation is presented as a multi-objective optimization problem. The univariate dimension reduction method is used to approximate the mean and variance of the design function. The non-dominated sorting genetic algorithm-II is used to solve the multi-objective optimization problem. To find the final optimum design from the Pareto front, Shannon's entropy-based technique for order of preference by similarity to ideal solution (TOPSIS) algorithm is applied. Finally, to demonstrate the applicability of the proposed method, two case studies are considered and the results are compared and discussed.

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1. Introduction

Engineering design inherently involves uncertainties (Arora 2004). Since deterministic constrained optimum design does not consider existing uncertainties, any violation due to uncertainties can decrease the reliability and quality of the system (Haldar and Mahadevan 2000). On the other hand, today's competitive commercial arena forces designers and manufactures to produce and supply low-price products with high quality at a desirable level of reliability. Therefore, designers must consider three objectives in the primary design stage: optimality, robustness and reliability. To achieve this aim, several general approaches have been proposed in the literature: robust design optimization (RDO), reliability-based design optimization (RBDO) and reliability-based robust design optimization (RBRDO).

While the objective of RDO is to enhance quality through minimizing performance variations due to uncertainties, the RBDO focuses on the feasibility of design constraints and rare events at the tail of their probability density functions (PDFs). In general, RDO and RBDO cannot guarantee the robustness and the reliability of a product simultaneously. Therefore, in recent years, RBRDO frameworks have been developed by incorporating RDO and RBDO approaches (Rathod *et al.* 2011). Using the eigenvector dimension reduction (EDR) method, Youn and Xi (2009) proposed an effective RBRDO methodology. The EDR method is an efficient and accurate probability analysis approach

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that facilitates sensitivity calculation by approximating the response surface using eigenvector samples. To design a robust and reliable product, Yadav, Bhamare, and Rathore (2010) proposed a hybrid quality loss function-based multi-objective optimization model. Yu, Gillot, and Ichchou (2013) developed a sensitivity-based sequential RBRDO to increase the efficiency of RBRDO. Wang, Li, and Savage (2015) developed a hybrid RBRDO approach by combining a single-loop approach and a mono-objective robust design. In general form, RBRDO is a multi-objective optimization problem with probabilistic constraints. To obtain reliable and robust Pareto-optimal points with different levels of reliability, Shahraki and Noorossana (2014) developed a general RBRDO methodology using an evolutionary multi-objective genetic algorithm. Considering the interaction of RDO and RBDO approaches, Lobato *et al.* (2020) proposed a new formulation to consider both robustness and reliability in the design procedure. To deal with dynamic uncertainties and handle time-dependent RBRDO problems, Zafar, Zhang, and Wang (2020) developed a multi-objective integrated framework. In another framework, developed by Libotte *et al.* (2020), a new formulation was proposed for multi-objective optimization problems to obtain solutions that are least sensitive to external noise and that satisfy prescribed reliability levels. To increase the efficiency of RBRDO, das Neves Carneiro and António (2021) developed an RBRDO approach by applying an analytical dimensional reduction technique of the uncertainty space associated with reliability assessment, which is based on the approximate local solution of Sobol' indices.

Almost all practical engineering problems include epistemic uncertainties, where most data sets for system uncertainties are insufficiently sampled from unknown statistical distributions. However, existing RBRDO methods have been developed based on the assumption that the random property of uncertain variables can be completely modelled using probability distributions, known as aleatory uncertainty. Therefore, the main focus of this article is to develop a novel RBRDO approach to deal with engineering design problems that involve both aleatory and epistemic uncertainties. Accordingly, to develop an integrated, efficient and accurate framework to deal with epistemic uncertainty for RBRDO, a frequentist method has been used for the robust design owing to its simplicity and lower computational cost, and a Bayesian-based approach is used for the reliability-based design because of its accurate results in reliability optimization. Using these methods provides the proposed approach with the ability to gradually update the degree of epistemic uncertainty about the problem by increasing the amount of sample data. The proposed framework fully addresses the amount of data available in the problem by representing incomplete information in terms of probability. In the proposed methodology, the original design problem under uncertainty is rewritten as an equivalent multi-objective optimization problem. Then, to obtain the optimum Pareto front, the non-dominated sorting genetic algorithm-II (NSGA-II), as a multi-objective optimization algorithm, is used. Finally, the entropy-based technique for order of preference by similarity to ideal solution (TOPSIS) is used to analyse the final results and select the most appropriate solution as the design point. Two engineering case studies under both aleatory and epistemic uncertainties are considered to demonstrate the capability of the proposed approach.

The remainder of this article is structured as follows. In Section 2, the proposed approach is explained in detail. Then, in Section 3, two mechanical examples are considered, and the obtained results are compared and discussed for verification. Finally, Section 4 concludes the article and discusses the advantages and limitations of the proposed approach.

2. Proposed Bayesian reliability-based robust design optimization (BRBRDO) method

The proposed approach consists of three main steps: (1) rewriting the design problem under uncertainty as a new multi-objective optimization problem; (2) using the NSGA-II method to solve the multi-objective optimization problem; and (3) selecting the best optimal solution according to TOPSIS.

In this section, the main steps of the proposed BRBRDO approach for design applications under both aleatory and epistemic uncertainties are explained.

2.1. Step 1: rewriting the design problem to the proposed RBRDO formulation

The general form of an RBRDO model can be expressed as (Shahraki and Noorossana 2014):

$$\begin{aligned}
&\min \quad f(\mu_{f(\boldsymbol{X},\boldsymbol{Y})}, \sigma_{f(\boldsymbol{X},\boldsymbol{Y})}) \\
&\text{s.t.} \quad P(g_j(\boldsymbol{X},\boldsymbol{Y}) \geq 0) \geq R_j, \quad j = 1,2,\ldots,nc \\
&\boldsymbol{\mu}_{\boldsymbol{X}}^L \leq \boldsymbol{\mu}_{\boldsymbol{X}} \leq \boldsymbol{\mu}_{\boldsymbol{X}}^U, \quad \boldsymbol{\mu}_{\boldsymbol{Y}}^L \leq \boldsymbol{\mu}_{\boldsymbol{Y}} \leq \boldsymbol{\mu}_{\boldsymbol{Y}}^U, \quad \boldsymbol{\mu}_{\boldsymbol{X}} = \{\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_{ndv}}\}, \quad \boldsymbol{\mu}_{\boldsymbol{Y}} = \{\mu_{Y_1}, \mu_{Y_2}, \ldots, \mu_{Y_{ndv}}\}
\end{aligned} \tag{1}$$

where $\boldsymbol{X}$ is the random variables vector, $\boldsymbol{Y}$ is the random parameters vector, $g_j(\boldsymbol{X},\boldsymbol{Y})$ denotes the jth limit state function [$(P(g_j(\boldsymbol{X},\boldsymbol{Y}) \geq 0) \geq R_j)$ is the jth probabilistic constraint function], $f(\boldsymbol{X},\boldsymbol{Y})$ is the cost function, and $\mu_{f(\boldsymbol{X},\boldsymbol{Y})}$ and $\sigma_{f(\boldsymbol{X},\boldsymbol{Y})}$ are its mean and variance, respectively. Quantities R_j, nc, ndv and ndp are the target reliabilities, number of probabilistic constraints, number of design variables and number of design parameters, respectively. The vectors $\boldsymbol{\mu}_{\boldsymbol{X}}$ and $\boldsymbol{\mu}_{\boldsymbol{Y}}$ represent the random variables mean vector and random parameters mean vector, respectively.

In this study, the proposed formulation as a multi-objective optimization problem contains three objectives: the performance function mean, the performance function variance as a robustness parameter, and the design reliability. It can be written as:

$$\min \quad \mu_f \tag{2}$$

$$\min \quad \sigma_f \tag{3}$$

$$\max \quad R_s \tag{4}$$

where μ_f is the performance function mean, σ_f is the performance function variance, and R_s is the reliability of the system.

2.2. Step 2: solving the proposed BRBRDO problem

The proposed RBRDO problem is formulated as a multi-objective optimization problem. Solving the multi-objective optimization problem presents a set of optimum solutions known as a Pareto front (also called non-dominated solutions). As a useful evolutionary algorithm to extract the optimal Pareto front(s) in multi-objective optimization problems, the elitist NSGA-II (Deb *et al.* 2002) is used in the proposed approach.

According to the proposed methodology, the objective functions should be evaluated repeatedly during the optimization process. The focus of this study is on incomplete information in the form of limited data, which is one of the most common types of epistemic uncertainty in engineering problems. Accordingly, two different methods are used in the proposed approach to quantify the existing epistemic uncertainties for various objective functions evaluations: the Bayesian reliability method for reliability evaluation, and the conservative frequentist approach proposed in Zaman *et al.* (2011) for evaluation of the performance function and robustness parameter. The uncertainty quantification methods are described in detail in the following subsections.

The following subsections explain how to approximate the objective functions in the presence of both epistemic and aleatory uncertainties.

2.2.1. Approximating the mean (objective 1) and variance (objective 2) of the performance function in the presence of both aleatory and epistemic uncertainties

In the proposed method, the univariate dimension reduction method (DRM) is used to approximate the mean and robustness of the performance function at each step of the optimization procedure. In the univariate DRM, it is supposed that PDFs of all variables and parameters are available. Therefore, to quantify the epistemic uncertainties in the performance function and robustness parameter, the conservative frequentist approach proposed in Zaman *et al.* (2011) is used. This method approximates the parameters of epistemic uncertainty distributions conservatively, as below.

Since the designer cannot control the epistemic design parameters, a gradient-based optimization method should be used to determine the conservative optimum values of the design parameters' means ($\mu^*_{P_s}$) at each optimization step (Zaman *et al.* 2011):

$$\begin{aligned} &\mu^*_{P_s} = arg_{\mu_{P_s}} \max\{w^* \mu_f(X^*, \mu_{P_s}) + (1-w)^* \sigma_f(X^*, \mu_{P_s})\} \\ &P_{S_l} \le \mu_{P_s} \le P_{S_u} \end{aligned} \tag{5}$$

where P_{S_l} and P_{S_u} are lower and upper limit vectors of the design parameters, respectively. X^* denotes the design variable vector at each design point and is considered constant in this optimization problem, and w is the weighting coefficient. Indeed, the optimum design variables ($\mu^*_{P_s}$) of the above formula are epistemic design parameters of the outer optimization process. Johnson's modified Student's t is used to construct the confidence bounds on the mean values of the design parameters in Equation (5). Thus, vectors P_{S_l} and P_{S_u} can be obtained as follows (Zaman *et al.* 2011):

$$\begin{aligned} P_{S_l} &= \overline{P_S} - t_{(\frac{\alpha}{2}, n-1)} \frac{S}{\sqrt{n}} + \frac{\mu_3}{6S^2 n} \\ P_{S_u} &= \overline{P_S} + t_{(\frac{\alpha}{2}, n-1)} \frac{S}{\sqrt{n}} + \frac{\mu_3}{6S^2 n} \end{aligned} \tag{6}$$

where $\overline{P_S}$ is the epistemic design parameters means vector, S is the sample standard deviation vector, n is the sample size of sparse points available for epistemic design parameters, ($t_{\alpha/2,n-1}$) is obtained from the Student's t distribution at degrees of freedom ($n-1$), and α is the significance level. Furthermore, μ_3 is the third central moment, which can be easily calculated as follows (Zaman *et al.* 2011):

$$\mu_3 = \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right)^3 \tag{7}$$

In the presence of epistemic uncertainty, the parameters of the PDF are also uncertain. Therefore, variances are assumed to follow the chi-squared distribution. Epistemic uncertainties of design variables and parameters are quantified through the two-sided ($1-\alpha$) confidence interval as (Zaman *et al.* 2011):

$$\left[\frac{(n-1)S^2}{c_{1-\frac{\alpha}{2}, n-1}}; \frac{(n-1)S^2}{c_{\frac{\alpha}{2}, n-1}} \right] \tag{8}$$

where n is the sample size, S is the sample standard deviation of sparse data points, $c_{1-\alpha/2,n-1}$ is obtained using the chi-squared distribution at ($n-1$) degrees of freedom, and α is the significance level. Once the variance intervals of the design variables and design parameters have been obtained, the upper bound variances are used to estimate the cost function variances using the univariate DRM.

In the univariate DRM, an n-dimensional function is decomposed to n one-dimensional functions. Then, a numerical integration method is applied to calculate the momentum integrals. Using the moment-based integration rule (MBIR) for an N-dimensional function $h(X) = h(x_1, x_2, \ldots, x_N)$, its mean and variance can be estimated as follows (Lee *et al.* 2008):

$$\begin{aligned} \mu_H \equiv E[H] &\cong E\left[\sum_{i=1}^{N} h(\mu_1, \ldots, \mu_{i-1}, x_i, \mu_{i+1}, \ldots, \mu_N) - (N-1) h(\mu_1, \ldots, \mu_N) \right] \\ &\cong \sum_{j=1}^{n} \sum_{i=1}^{N} w_i^j h(\mu_1, \ldots, \mu_{i-1}, x_i^j, \mu_{i+1}, \ldots, \mu_N) - (N-1) h(\mu_1, \ldots, \mu_N) \end{aligned} \tag{9}$$

$$\begin{aligned}\sigma_H^2 &\equiv E[h(X)-\mu_H^2]\\ &\cong E\left[\sum_{i=1}^{N} h^2(\mu_1,\ldots,\mu_{i-1},x_i,\mu_{i+1},\ldots,\mu_N)-(N-1)h^2(\mu_1,\ldots,\mu_N)\right]-\mu_H^2\\ &\cong \sum_{j=1}^{n}\sum_{i=1}^{N} w_i^j h^2(\mu_1,\ldots,\mu_{i-1},x_i^j,\mu_{i+1},\ldots,\mu_N)-(N-1)h^2(\mu_1,\ldots,\mu_N)-\mu_H^2 \end{aligned} \tag{10}$$

where μ_H and σ_H^2 are the mean and variance of the performance function, respectively, μ_i is the mean value of a random variable x_i, n is the number of quadrature points, and w_i^j is the weighting coefficient of the jth quadrature point.

2.2.2. Reliability evaluation in the presence of both aleatory and epistemic uncertainty

As an objective function, the design reliability should be evaluated at each design point during the optimization process. In this study, the reliability of the system (design reliability) is approximated simply as the minimum of the performance criteria reliabilities. Accordingly, the reliability of all constraints should be evaluated at each design point. Then, among all the constraints' reliabilities, the lowest value is considered as the design reliability.

In the presence of epistemic uncertainty in the form of sparse data points, reliability is itself an uncertain parameter. Therefore, to approximate the distribution of the reliability of constraints that contain epistemic design variables/parameters, Bayesian reliability analysis (Gunawan and Papalambros 2006) is utilized. The Bayesian reliability analysis method is explained in detail in Subsection 2.2.2.1. However, for constraints that contain only aleatory uncertainty, the reliability can be evaluated using any reliability analysis method. In this study, the first order reliability method (FORM) method is used. FORM is explained in brief in Subsection 2.2.2.2.

2.2.2.1. Bayesian reliability method. To approximate the distribution of the reliability of constraints including epistemic uncertainty, the Bayesian reliability method uses the Bayesian inference problem. Accordingly, the reliability of a constraint, $Pr[P_{g_j}(0)]$, can be estimated with a beta distribution, as follows:

$$\Pr[P_{g_j}(0)] = Beta(P_{g_j}(0); E_j(r)+1, N-E_j(r)+1) \tag{11}$$

where N is the sample size of epistemic uncertainties, and $E_j(r)$, which is defined as the expected total number of feasible realizations of the design, can be calculated as follows:

$$E_j(p) = \sum_{k=1}^{N} Pr[g_j(X_t,P_t)\le 0|(X_s,P_s)_k] \tag{12}$$

where X_t is the aleatory design variables vector, P_t is the aleatory design parameters vector, and X_s and P_s are the epistemic design variables vector and the epistemic design parameters vector, respectively.

Because of incomplete information, reliability is an uncertain parameter and only an estimate of the design reliability distribution can be obtained, instead of a certain value. In such cases, defining the confidence interval of the probability that the target reliability is equal to or greater than the corresponding certain value, $\xi_j = Pr[P_{g_j}(0)|_{\mu_X} \ge R_j]$, the reliability can be obtained as follows:

$$R_j = 1 - \mathrm{CDF}[1-\xi_j];\ j = 1,2,\ldots,J \tag{13}$$

where CDF is the cumulative density function and J is the total number of constraints. It can be inferred from this formula that for $\xi_j = 1$, the design is certainly reliable ($R_j = 1$), while in the case where $\xi_j = 0$, the design is definitely unreliable.

2.2.2.2. Reliability analysis algorithm: FORM. In the proposed algorithm, FORM (Der Kiureghian 2005) has been applied to evaluate the reliability at each step of the optimization process. In this method, the limit state function is linearized using the first order series expansion. This linearization is carried out at the most probable point (MPP) in the standard normal space. The MPP, which is the nearest point to the origin on the limit state function, can be determined through an optimization problem as follows:

$$y^* = argmin\{\tilde{y}|G(\tilde{y}) = 0\} \tag{14}$$

where $G(\tilde{y})$ is the limit state function. To solve the optimization problem in Equation (14) without any convergence problem, the improved Hasofer–Lind–Rackwitz–Fiessler (HLRF) algorithm (Zhang and Kiureghian 1995), which is one of the newest optimization tools, is used in the proposed approach. Finding the MPP, the design reliability can be evaluated as follows:

$$\beta = y^* \rightarrow R = 1 - \varphi(-\beta) \tag{15}$$

where β is the reliability index, φ is the CDF of the standard normal distribution, and R is the reliability.

2.2.2.3. Design reliability evaluation algorithm. The flowchart of the design reliability evaluation can be summarized as follows:

For each constraint:

- *If the constraint contains design variables and parameters that include epistemic uncertainty, then:*
  1. In the first step, the confidence level ξ should be determined for each constraint. It is assumed that ξ is determined and then the reliability value is maximized as an objective function.
  2. The reliability of each constraint should be evaluated for each sample set of epistemic samples $((X_s, P_s)_k)$ through the reliability analysis.
  3. The expected total number of feasible realizations $E_j(p)$ should be obtained using Equation (12).
  4. For each constraint, the posterior distribution should be estimated using Equation (11).
  5. After obtaining the reliability distribution, a crisp value of reliability can be obtained using Equation (13).
- *If the constraint does not contain any epistemic uncertainty, then:* Reliability can be evaluated using FORM directly.
- After evaluating the reliability of all constraints, the lowest value of the obtained reliability values, as the design reliability, can be approximated as:

$$R_s = \min(R_j), \quad j = 1, 2, \ldots, J \tag{16}$$

where J is the number of constraints and R_s is the design reliability.

2.3. Step 3: selecting the final optimum design from the Pareto front

In RBRDO problems, selecting the best solution from the optimal Pareto front(s) is usually one of the challenges of multi-criteria decision making. To select the best design from the optimal Pareto solutions, the designer should specify the importance of objectives and the corresponding preferences, which is not a simple procedure. Therefore, in the proposed approach, the integrated Shannon's entropy-based TOPSIS is used to select the best optimal design variables without the weighting procedure (Deng, Yeh, and Willis 2000). According to some previous applications in the literature (Khodaygan 2019; Ghaderi, Hassani, and Khodaygan 2021), the best solution from the optimal Pareto solutions can be selected by the enhanced Shannon's entropy-based TOPSIS through the following steps:

1. Creating a matrix consisting of m rows of alternatives and n columns for objectives:

$$A = [a_{ij}]_{m \times n} \tag{17}$$

2. Normalizing matrix A:

$$B = [b_{ij}]_{m \times n} \tag{18}$$

where b_{ij} is determined as

$$b_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^{m} a_{kj}^2}} \tag{19}$$

3. Normalizing the decision matrix with respect to objective $f_j (j = 1, 2, \ldots, n)$.
4. Obtaining the projection value (p_{ij}):

$$p_{ij} = \frac{b_{ij}}{\sum_{i=1}^{m} b_{ij}} \tag{20}$$

5. Calculating the entropy values e_j for each criterion f_j:

$$e_j = \frac{-1}{\ln(m)} \sum_{i=1}^{m} p_{ij} \ln p_{ij} \tag{21}$$

6. Calculating the divergence degree of each criterion f_j:

$$d_j = 1 - e_j \tag{22}$$

The value of d_j means that the criterion f_j is more important for selection.

7. The weights of the objective f_j can be determined by:

$$w_j = \frac{d_j}{\sum_{k=1}^{n} d_k} \tag{23}$$

Then, the normalized weighted evaluation matrix is obtained as

$$U = [u_{ij}]_{m \times n} = [w_i b_{ij}]_{m \times n} \tag{24}$$

8. Selecting the positive ideal solution (PIS) (A^+) and the negative ideal solution (NIS) (A^-):

$$\begin{aligned} A^+ &= \{u_1^+, u_2^+, \ldots u_n^+\} \\ A^- &= \{u_1^-, u_2^-, \ldots u_n^-\} \end{aligned} \tag{25}$$

9. Computing the distance of candidate i from the PIS and NIS:

$$\begin{aligned} d_i^+ &= \left\{ \sum_{j=1}^{n} (u_{ij} - u_j^+)^2 \right\}^{1/2} \\ d_i^- &= \left\{ \sum_{j=1}^{n} (u_{ij} - u_j^-)^2 \right\}^{1/2} \end{aligned} \tag{26}$$

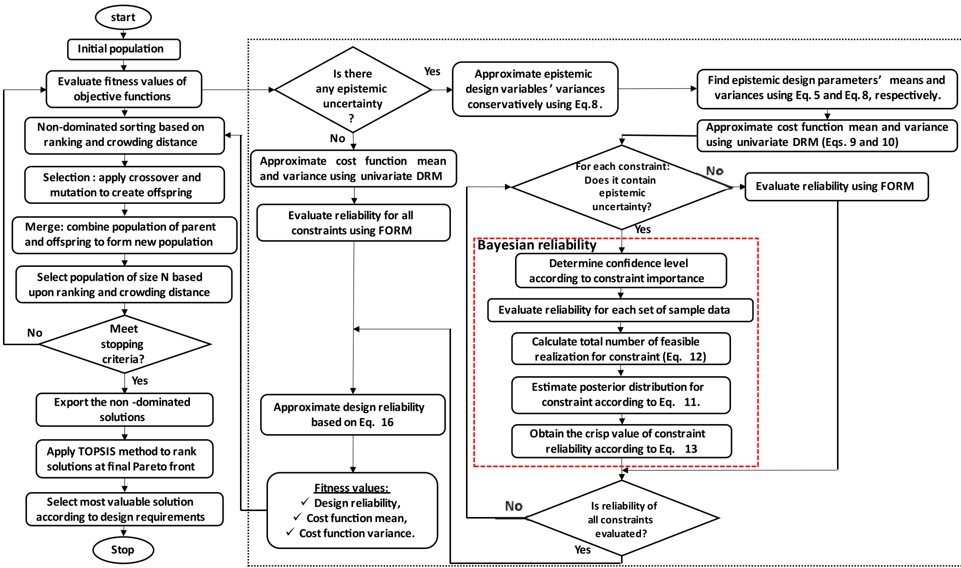

Figure 1. Flowchart of the proposed Bayesian reliability-based robust design optimization (BRBRDO) algorithm. TOPSIS = technique for order of preference by similarity to ideal solution; DRM = dimension reduction method.

10. Determining the relative distance (the closeness coefficient) of candidate i for the PIS:

$$C_i^* = \frac{d_i^-}{d_i^- + d_i^+} \tag{27}$$

A high value of C_i^* means that the relative distance is closer to the PIS and it is equal to a better rank. Consequently, all candidates on the optimal Pareto front should be sorted with respect to the corresponding closeness coefficients (C_i^*).

2.4. Flowchart of the proposed algorithm

For clarification and to summarize the steps of the proposed BRBRDO algorithm, the flowchart of the proposed algorithm is shown in Figure 1.

3. Case studies

In this section, to demonstrate the applicability of the proposed method, two case studies are considered. The computational results of the proposed BRBRDO method are compared to the results obtained by other existing approaches in the literature.

3.1. Case study 1: tension rod problem

As the first case study, a tension bar in the automobile suspension system is considered (Figure 2) (Zhang 2015).

In this study, the weight of the tension bar should be minimized, and the inner diameter d_i and outer diameter d_o are considered as design variables with epistemic and aleatory uncertainties, respectively. Since the density of the specified material of the tension rod is constant, its area (A) in a specific length represents the rod mass. The geometric limitation and the criterion to prevent structural failure

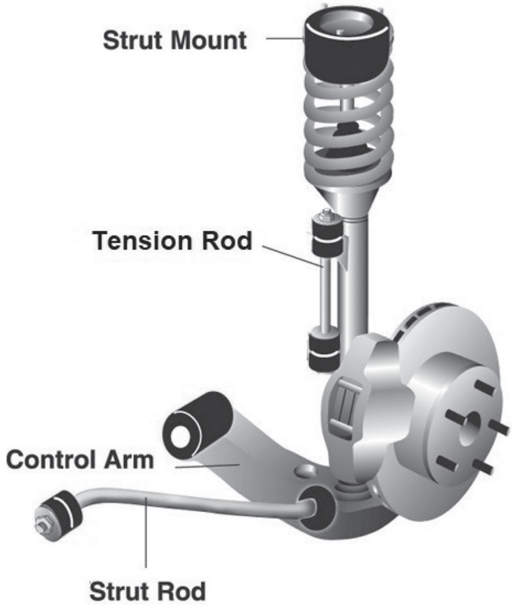

Figure 2. Tension rod of the automobile suspension system.

Table 1. Specification of the design variables in the tension rod.

Design variable	Probability distribution	Uncertainty	Standard deviation(mm)	Lower limit(mm)	Upper limit(mm)
d_o	Normal	Aleatory	3.5	30	40
d_i	Normal	Epistemic	2.5	20	30

Table 2. Specification of the design parameters in the tension rod.

Design parameter	Probability distribution	Uncertainty	Mean	Standard deviation
F	Normal	Aleatory	170 (N)	2.6 (N)
S	Normal	Aleatory	400 (MPa)	11 (MPa)

as the constraints can be formulated as follows:

$$\begin{aligned} -d_o + d_i &\leq 0 \\ \frac{4F}{\pi({d_o}^2 - {d_i}^2)} - S &\leq 0 \end{aligned} \tag{28}$$

where F and S are the tensile load and the material tensile strength, respectively. Accordingly, the optimization problem can be formulated as:

$$\text{minimize}_X \; f(X) = \frac{\pi}{4}({d_o}^2 - {d_i}^2) \tag{29}$$

$$\text{subject to}: g_1(X) = -d_o + d_i \leq 0, \quad g_2(X,P) = \frac{4F}{\pi({d_o}^2 - {d_i}^2)} - S \leq 0$$

The design variables are divided into epistemic design variables $X_s = \{d_i\}$ and aleatory design variables $X_t = \{d_o\}$, and the design parameters also include aleatory uncertainties; $P_t = \{F, S\}$. All uncertain variables and parameters are distributed normally according to Tables 1 and 2, respectively.

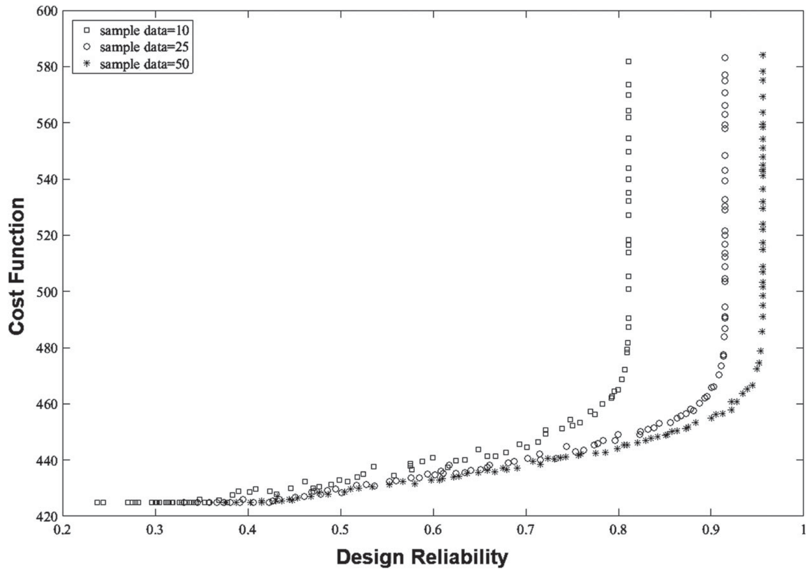

Figure 3. Effect of epistemic uncertainty on the obtained Pareto front of the design reliability and the cost function in the tension rod design problem.

To analyse the effects of epistemic uncertainty on objective functions, the optimization problem can be solved at three levels for the number of samples ($N = 10$, 25 and 50) of the design variable d_i with epistemic uncertainty ($\xi = 0.9$ and $\alpha = 0.05$). Using the proposed algorithm, optimal Pareto fronts are shown in Figures 3–5. As shown in Figure 3, the design reliability is increased by increasing the number of data points available. In other words, at the desired level of reliability, the cost function mean will be decreased by decreasing epistemic uncertainty. In such a case, maximum reliability can be achieved in situations where the sample size is large enough that epistemic uncertainty can be described through the specific probability distribution. In the presence of epistemic uncertainty, the variance is an uncertain parameter. The uncertainty can be reduced by increasing the number of samples or decreasing the confidence value (α). Figure 4 shows the effects of epistemic uncertainty on the cost function mean and variance trade-off. According to Figure 4, the variation of the variance is decreased by increasing the number of samples. In the case where all uncertainties are aleatory, the variation of the variance is the minimum.

Figure 5 shows the epistemic uncertainty effect on the obtained Pareto front of the design reliability and the variance. According to Figure 5, increasing N will lead to an improvement in the obtained optimal Pareto front. In other words, decreasing epistemic uncertainty leads to an optimal design with higher reliability, lower variance and the lower cost function mean in the Pareto fronts (see Figures 3–5).

To analyse the effect of the confidence level of the design reliability on the obtained optimal results, the optimization problem is solved at three confidence levels (75% or $\xi = 0.25$, 90% or $\xi = 0.10$ and 99% or $\xi = 0.01$). The obtained optimum Pareto fronts are illustrated in Figure 6. Referring to Figure 6, at constant design reliability, the cost function mean increases by increasing the confidence level of the design reliability (ξ).

To achieve the optimal design with the best values of objective functions, the proposed method is carried out at $\xi = 0.9$ and $n = 25$. According to these settings, the three-dimensional (3D) optimal Pareto front is shown in Figure 7.

The proposed framework is a double-loop optimization problem in which the reliability evaluation algorithm is nested within the optimization loop. Therefore, the number of function evaluations

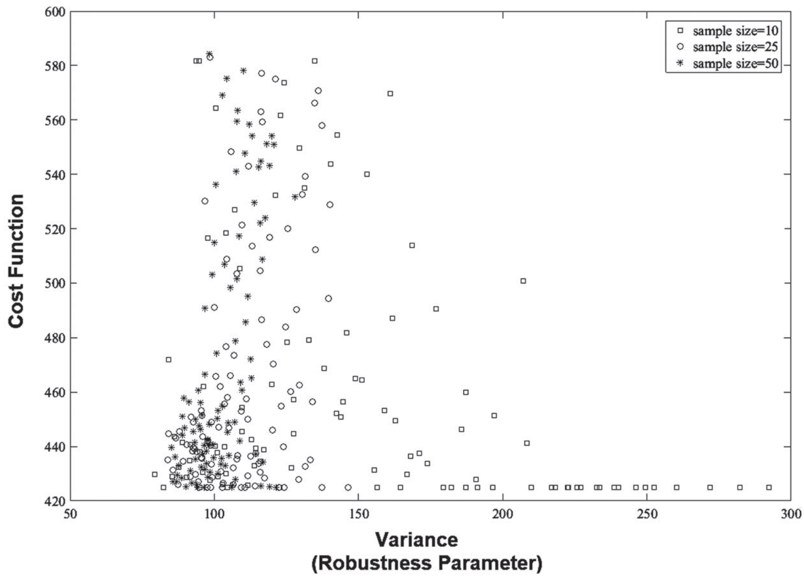

Figure 4. Effect of epistemic uncertainty on the obtained Pareto front of the variance (robustness parameter) and the cost function in the tension rod design problem.

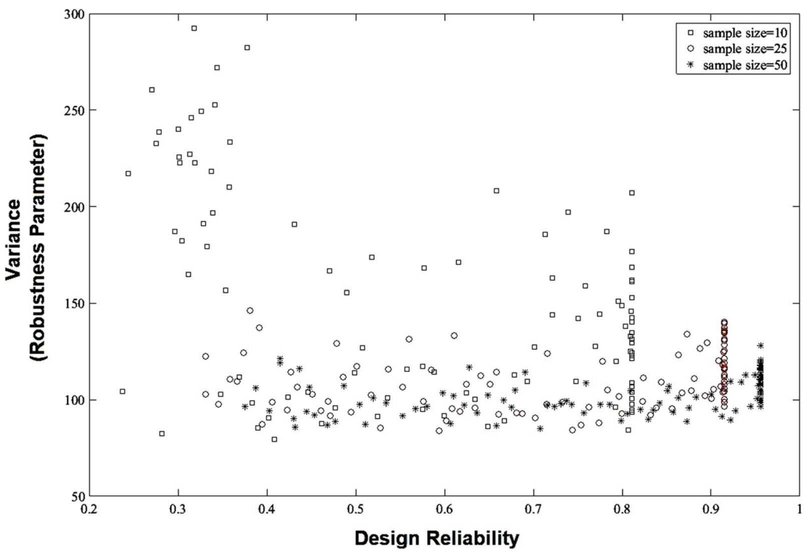

Figure 5. Effect of epistemic uncertainty on the obtained Pareto front of the design reliability and the variance (robustness parameter) in the tension rod design problem.

and the calculation time are relatively high. Numerous parameters affect the calculation time in each problem. The most important parameters are the sample points of epistemic uncertainty, number of constraints, number of design variables and design parameters, population size, number of generations and nonlinearity of constraints. To solve this problem, the proposed algorithm, which was written in MATLAB®, was run on a 1.8 GHz processor with 4 GB of RAM. The running time and number of function evaluations are reported in Table 3. In this table, the number of goal function evaluations that the DRM needs to approximate cost function mean and variance, and the number of

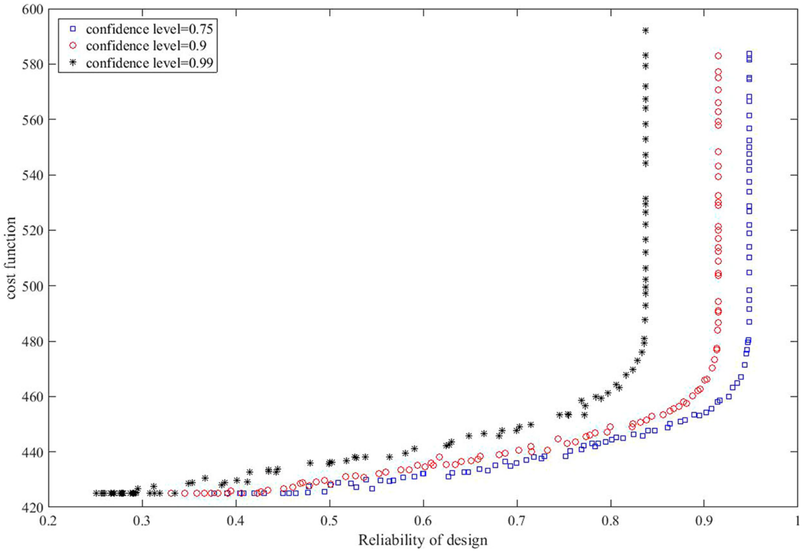

Figure 6. Confidence level effect on the Pareto front of the cost function and the design reliability in the tension rod design problem.

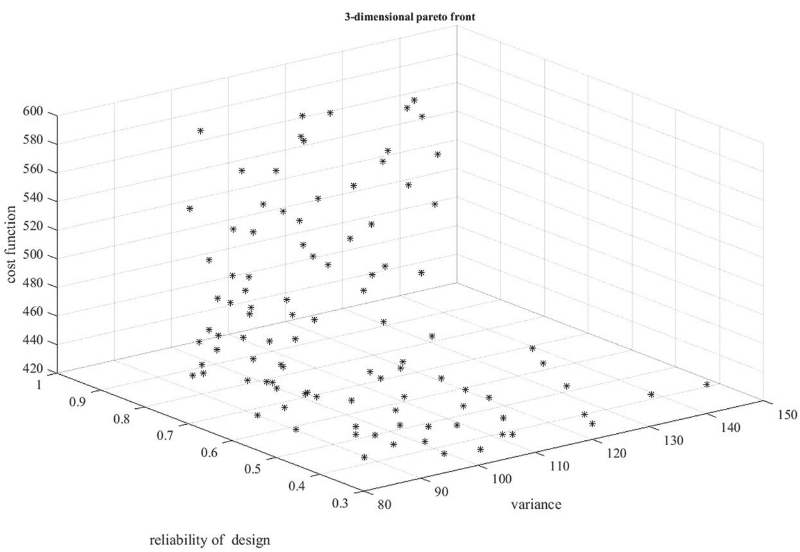

Figure 7. Three-dimensional trade-off frontier obtained for the tension rod design problem with $n = 25$ and $\xi = 0.9$.

goal functions evaluated in the reliability evaluation loop are reported separately. It should be noted that any reliability analysis method can be used in the proposed approach. Hence, other efficient reliability analysis methods, such as the EDR method (Youn, Xi, and Wang 2008), can be used to decrease the computational efforts of the proposed approach.

To select the best solution from the 3D Pareto front, Shannon's entropy-based TOPSIS method is used to rank the optimal design points (Figure 8). After ranking the optimal design points, optimal solution 54 with the highest relative distance value ($C^* = 0.997$) is found to be the closest optimal design point to the ideal solution (reliability of system $= 0.87$, cost function mean $= 491\ \text{mm}^2$, variance $= 99\ (\text{mm}^2)^2$).

Table 3. Number of function evaluations and run time needed in the proposed approach to solve the tension rod problem.

Number of function evaluations			
Dimension reduction method	Reliability evaluation	Total	Run time (s)
$300 \times 100 \times 5 = 150{,}000$	20,250,000	20,400,000	3.78233E3

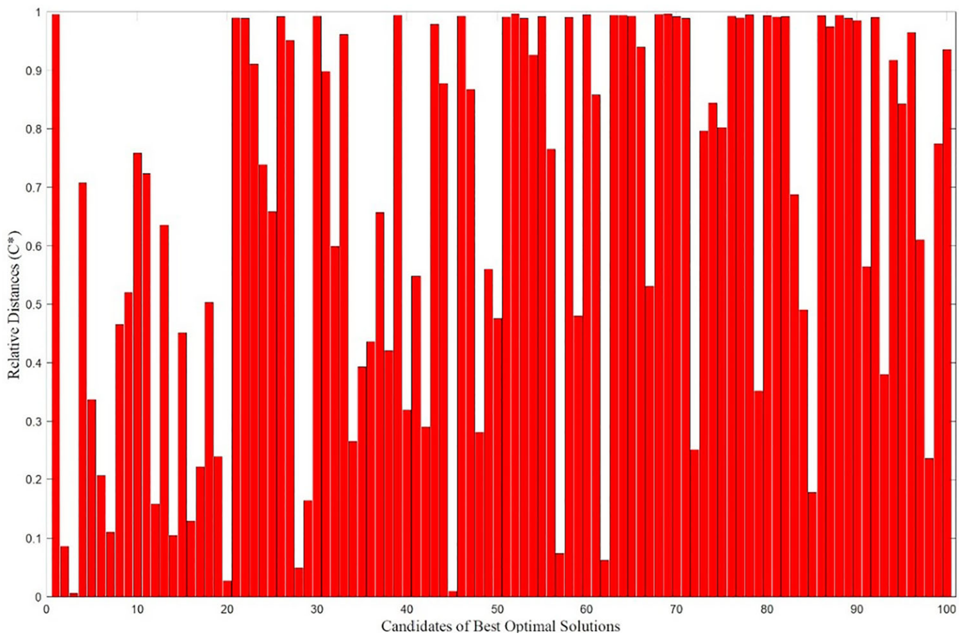

Figure 8. Ranking the optimal design points on the three-dimensional Pareto front (tension rod problem).

Table 4. Comparison of the obtained optimum design using the proposed method and other approaches in the tension rod problem.

Approach	d_o (mm)	d_i (mm)	Objective 1: area	Objective 2: variance	Objective 3: reliability
DO	34.781	25.857	425.02	117.19	0.50
RDO	30.877	20.128	430.61	85.21	0.51
BRBDO	34.233	24.533	448.25	118.52	0.81
Proposed algorithm (BRBRDO)	32.973	21.495	491.00	99.00	0.87

Note: DO = deterministic optimization; RDO = robust design optimization; BRBDO = Bayesian reliability-based design optimization; BRBRDO = Bayesian reliability-based robust design optimization.

To verify the proposed framework, the design point obtained using the proposed approach is compared with three approaches from the literature, as reported in Table 4. For this problem, the parameters of NSGA-II are adjusted as follows: population size = 100 individuals, generation number = 300 generations, crossover rate = 0.9 and mutation rate = 0.2.

According to Table 4, deterministic optimization (DO) minimizes the cost function regardless of the existing uncertainties. So, the optimal solution obtained by the DO approach has the minimum value of the cost function without considering the variance of the cost function and the reliability. Therefore, the solution obtained by this method has a high value of the variance of the cost function due to uncertainties at a very low level of reliability, which is unacceptable. The proposed method simultaneously optimizes three objectives (*i.e.* the cost function, robustness parameter and system reliability). Therefore, the solution obtained by the proposed approach may increase the cost function to reduce the variance of the objective function and increase reliability. According to Table 4,

Table 5. Relative distance to **IS*** of the proposed approach and Bayesian reliability-based design optimization (BRBDO) solutions in the tension rod problem.

Approach	R_D
Proposed approach	0.2243
BRBDO	0.4007

the proposed approach presents a design with 15.52% lower variance and 74% higher reliability in comparison to the solution obtained by the DO method.

Referring to Table 4, in the RDO approach, the cost function mean and variance have been minimized simultaneously (using the weighted sum method as a single-objective optimization approach), where the reliability of the system has been ignored in the design considerations. So, even though the RDO approach may obtain a solution with lower values for the cost function and variability, it fails to provide solutions with an acceptable level of reliability. The proposed approach can find an optimal solution with an acceptable level of reliability by making trade-offs between the objective functions. According to Table 4, the proposed approach presents a solution with a higher reliability value, by 72%, compared to the RDO approach.

Consequently, the results demonstrate that DO and RDO fail to ensure the reliability of the design, and in uncertain conditions, these two approaches lead to product failure. In such conditions, only the third approach seems to be comparable with the proposed approach. In the third approach, which is Bayesian reliability-based design optimization (BRBDO) (Srivastava and Deb 2013), the Bayesian reliability concept has been applied to evaluate the reliability in the presence of epistemic uncertainties. Although this approach increases reliability by increasing the cost function, it fails to consider the robustness in the design process in the existence of epistemic uncertainty. According to Table 4, the reliability and the variance of the design obtained by the proposed method, in comparison to the BRBDO method, have been improved by 7.4% and 16.47%, respectively, while the cost function value has been increased by about 9.5%. To compare the proposed method and BRBDO approach, a non-achievable solution is defended as an ideal solution (IS^*), in which the corresponding values of the three objectives are simultaneously best (*i.e.* $IS^* = [IS_1^*, IS_2^*, IS_3^*]$). The relative distance of the obtained objectives (*i.e.* $F^* = [F_1^*, F_2^*, F_3^*]$) from the ideal solution (IS^*), which is non-achievable, is a good measure that demonstrates the superiority of the proposed approach over the BRBDO approach. The relative distance to IS^* (R_D) can be calculated as follows:

$$R_D = \left[\left(\frac{F_1^* - IS_1^*}{IS_1^*} \right)^2 + \left(\frac{F_2^* - IS_2^*}{IS_2^*} \right)^2 + \left(\frac{F_3^* - IS_3^*}{IS_3^*} \right)^2 \right]^{0.5} \tag{30}$$

where F_i^* is the ith objective value of the solution, and IS_i is the ith objective value of the ideal solution. The R_D values of the proposed approach solution and BRBDO solution to ($IS^* = (425.02, 85.21, 0.87)$ are reported in Table 5. The optimal objectives obtained by the proposed approach, compared to the BRBDO method, are closer to the ideal solution (IS^*).

3.2. Case study 2: side-impact test problem

The side-impact test problem according to the European Enhanced Vehicle-safety Committee (EEVC) is considered as the second case study. In this study, the optimization procedure can be carried out to improve the dummy safety performance while minimizing the weight (Srivastava and Deb 2013). In the side-impact test, the head injury criterion (HIC), abdomen load, pubic symphysis force, viscous criteria (VC) and rib deflections (upper, middle and lower) are considered as the main specifications of responses of the dummy. The side-impact test problem is formulated as follows (Srivastava and Deb 2013):

$$
\begin{aligned}
&\text{minimize}_X f(\boldsymbol{X}): \ \text{Weight} = 1.98 + 4.9X_1 + 6.67X_2 + 6.98X_3 + 4.01X_4 + 1.78X_5 \\
&\quad + 0.00001X_6 + 2.73X_7 \\
&\text{subject to: } g_1(\boldsymbol{X},\boldsymbol{P}): \text{Abdomen load (kN)} \\
&\quad = 1.16 - 0.3717X_2X_4 - 0.00931X_2X_{10} - 0.484X_3X_9 \\
&\qquad + 0.01343X_6X_{10} \le 1 \\
&g_2(\boldsymbol{X},\boldsymbol{P}): \text{Viscous criteria}_{\text{upper}}(\text{ms}^{-1}) \\
&\quad = 0.261 - 0.0159X_1X_2 - 0.188X_1X_8 - 0.019X_2X_7 \\
&\qquad + 0.0144X_3X_5 + 0.0008757X_5X_{10} + 0.08045X_6X_9 + 0.00139X_8X_{11} \\
&\qquad + 0.00001575X_{10}X_{11} \le 0.32 \\
&g_3(\boldsymbol{X},\boldsymbol{P}): \ \text{Viscous criteria}_{\text{middle}}(\text{ms}^{-1}) \\
&\quad = 0.214 + 0.00817X_5 - 0.131X_1X_8 - 0.0704X_1X_9 + 0.03099X_2X_6 - 0.018X_2X_7 \\
&\qquad + 0.0208X_3X_8 + 0.121X_3X_9 - 0.00364X_5X_6 + 0.000771X_5X_{10} - 0.0005354X_6X_{10} \\
&\qquad + 0.00121X_8X_{11} + 0.00184X_9X_{10} - 0.018X_2^2 \le 0.32 \qquad (31) \\
&g_4(\boldsymbol{X},\boldsymbol{P}): \ \text{Viscous criteria}_{\text{lower}}(\text{ms}^{-1}) \\
&\ = 0.74 - 0.61X_2 - 0.163X_3X_8 + 0.001232X_3X_{10} \\
&\qquad - 0.166X_7X_9 + 0.227X_2^2 \le 0.32 \\
&g_5(\boldsymbol{X},\boldsymbol{P}): \text{Upper rib deflection (mm)} \\
&\quad = 28.98 + 3.818X_3 - 4.2X_1X_2 + 0.0207X_5X_{10} \\
&\qquad + 6.63X_6X_9 - 7.77X_7X_8 + 0.32X_9X_{10} \le 32 \\
&g_6(\boldsymbol{X},\boldsymbol{P}): \ \text{Middle rib deflection (mm)} \\
&\quad = 33.86 + 2.95X_3 + 0.1792X_{10} - 5.057X_1X_2 - 11X_2X_8 \\
&\qquad - 0.0215X_5X_{10} - 9.98X_7X_8 + 22X_8X_9 \le 32 \\
&g_7(\boldsymbol{X},\boldsymbol{P}): \ \text{Lower rib deflection (mm)} \\
&\quad = 46.36 - 9.9X_2 - 12.9X_1X_8 + 0.1107X_3X_{10} \le 32 \\
&g_8(\boldsymbol{X},\boldsymbol{P}): \ \text{Pubic force (kN)} \\
&\quad = 4.72 - 0.8X_4 - 0.19X_2X_3 - 0.0122X_4X_{10} \\
&\qquad + 0.009325X_6X_{10} + 0.000191X_{11}^2 \le 4 \\
&g_9(\boldsymbol{X},\boldsymbol{P}): \ \text{Velocity of B-pillar at the middle point (mm/ms)} \\
&\quad = 10.58 - 0.674X_1X_2 - 1.95X_2X_8 + 0.02054X_3X_{10} \\
&\qquad - 0.0198X_4X_{10} + 0.028X_6X_{10} \le 9.9 \\
&g_{10}(\boldsymbol{X},\boldsymbol{P}): \ \text{Velocity of the front door at the B - pillar (mm/ms)} \\
&\quad = 16.45 - 0.489X_3X_7 - 0.843X_5X_6 + 0.0432X_9X_{10} \\
&\qquad - 0.0556X_9X_{11} - 0.000786X_{11}^2 \le 15.7
\end{aligned}
$$

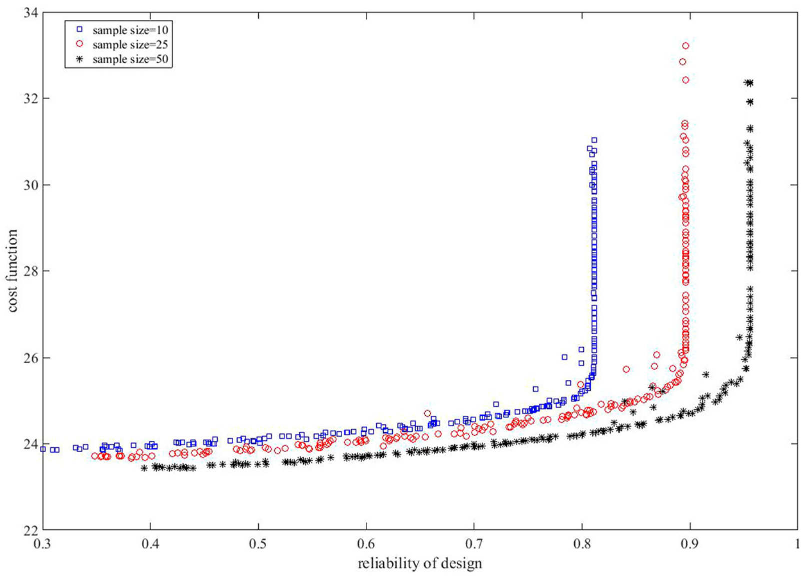

Figure 9. Effect of epistemic uncertainty on the obtained Pareto front of the design reliability and the cost function in the car side-impact test problem.

Table 6. Description of design parameters in the side-impact test problem.

Design parameter	Probability distribution	Uncertainty	Mean	Standard deviation
X_8	Normal	Epistemic	0.345	0.006
X_9	Normal	Aleatory	0.192	0.006
X_{10}	Normal	Aleatory	0	10
X_{11}	Normal	Aleatory	0	10

Table 7. Description of design variables in the car side-impact test.

Design variable	Probability distribution	Uncertainty	Standard deviation	Lower bound	Upper bound
X_1	Normal	Aleatory	0.03	0.5	1.5
X_2	Normal	Aleatory	0.03	0.5	1.5
X_3	Normal	Aleatory	0.03	0.45	1.35
X_4	Normal	Aleatory	0.03	0.5	1.5
X_5	Normal	Epistemic	0.03	0.875	2.625
X_6	Normal	Aleatory	0.03	0.4	1.2
X_7	Normal	Aleatory	0.03	0.4	1.2

where $f(\boldsymbol{X})$ is weight, $\{g_1(\boldsymbol{X},\boldsymbol{P}),\ldots,g_8(\boldsymbol{X},\boldsymbol{P})\}$ are constraints that are defined on the dummy's response, and $\{g_9(\boldsymbol{X},\boldsymbol{P}),g_{10}(\boldsymbol{X},\boldsymbol{P})\}$ are the car-related constraints. It is assumed that epistemic uncertainties exist in both design variables and design parameters. So, design variables are divided into $\boldsymbol{X}_s=\{X_1,X_5\}$ and $\boldsymbol{X}_t=\{X_2,X_3,X_5,X_6,X_7\}$ and design parameters are divided into $\boldsymbol{P}_s=\{X_8\}$ and $\boldsymbol{P}_t=\{X_9,\ldots,X_{11}\}$. Design parameters and design variables are defined in Tables 6 and 7, respectively.

For analysis of the epistemic uncertainty effect on the objective functions, the problem is solved under different sample sizes ($n=10, 25$ and 50) based on the available data. Accordingly, the obtained Pareto fronts are shown in Figures 9–11. By increasing the sample size, the design reliability increases at the desired value of the cost function. The maximum reliability that can be reached at the desired value of the confidence level is also increased by increasing the sample size. In other words, by increasing information about the design variables that are involved with the epistemic uncertainty, the design reliability at the specific desired confidence level is improved.

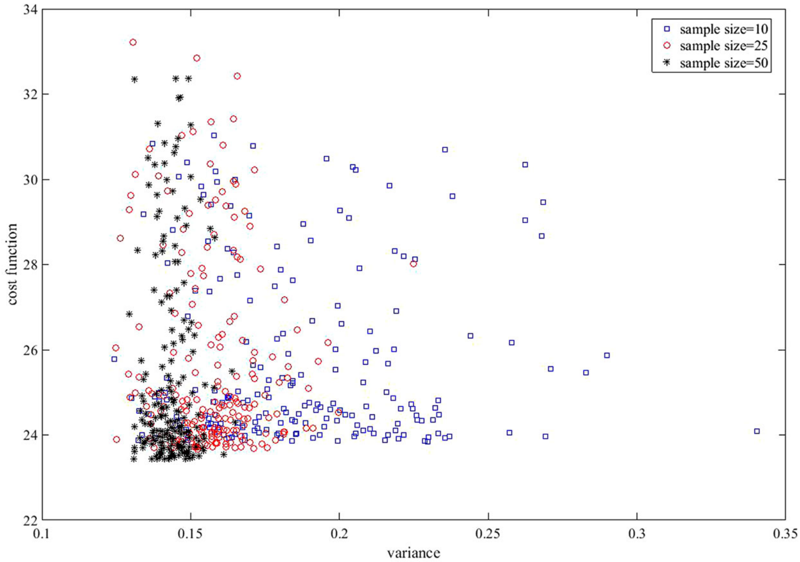

Figure 10. Effect of epistemic uncertainty on the obtained Pareto front of the variance (robustness parameter) and the cost function in the car side-impact test problem.

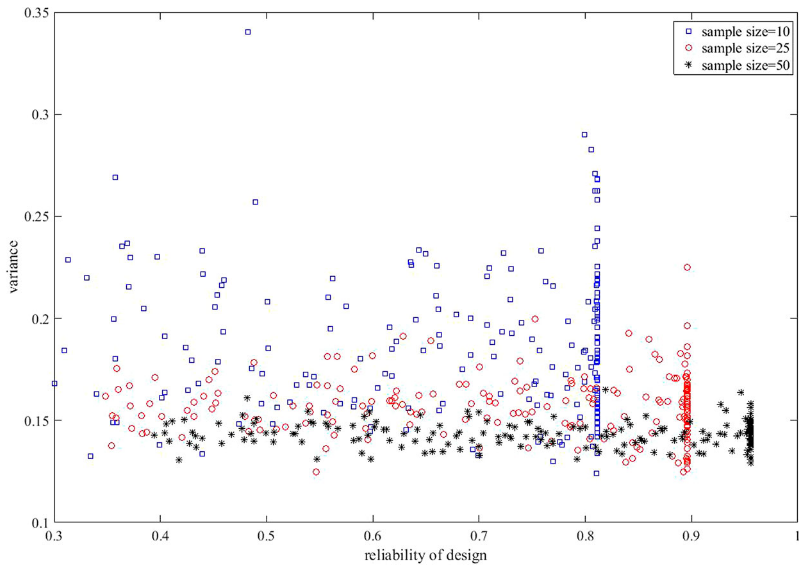

Figure 11. Effect of epistemic uncertainty on the obtained Pareto front of the design reliability and the variance (robustness parameter) in the car side-impact test problem.

To analyse the effect of confidence level on the objective functions, the problem is solved by the proposed algorithm under the different confidence levels: 0.75, 0.90 and 0.99. By reducing ξ (or increasing the confidence level), the design reliability is increased at the desired value of the cost function (Figure 12).

To achieve the best values for the objective functions, this problem with $\xi = 0.9$ and $n = 25$ is solved using the proposed algorithm, and the 3D optimal Pareto front is shown in Figure 13.

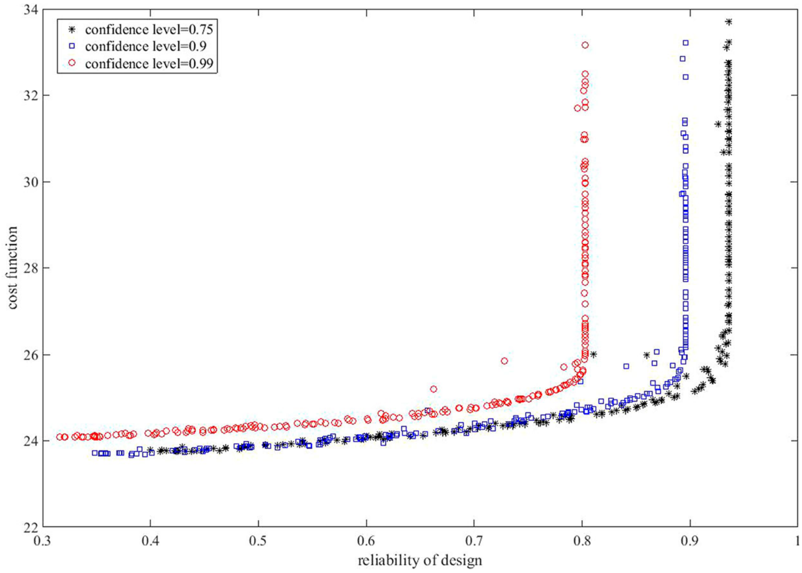

Figure 12. Confidence level effect on the cost function and the design reliability trade-off in the car side-impact test problem.

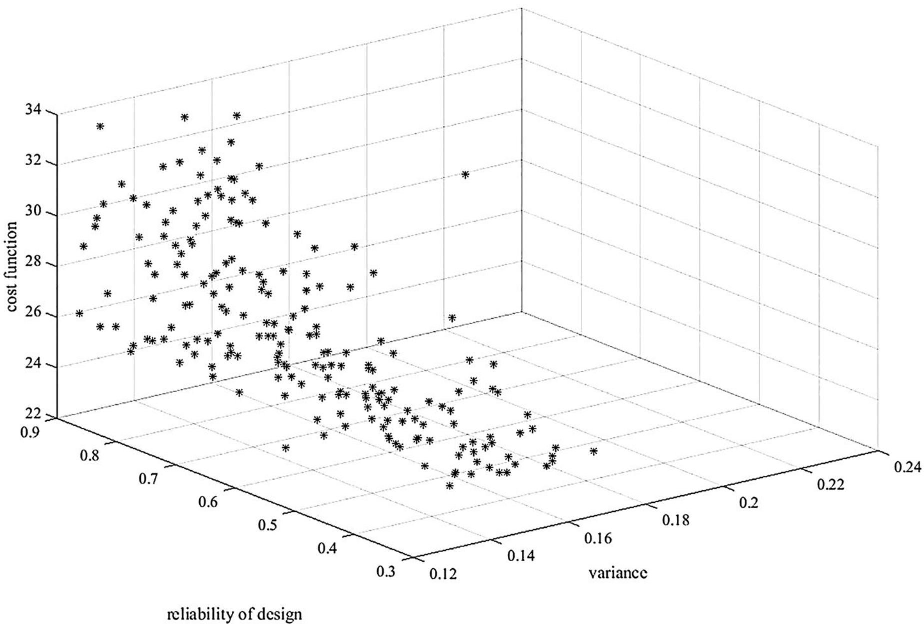

Figure 13. Three-dimensional trade-off frontier obtained for the car side-impact test problem with $n = 25$ and $\xi = 0.9$.

The proposed approach was run on a 1.8 GHz processor with 4 GB of RAM to solve this problem, and the calculation time and the number of function evaluations needed in the proposed approach to solve this problem are reported in Table 8.

Finally, to select the best solution from the 3D Pareto front (Figure 13), Shannon's entropy-based TOPSIS method is used to rank the optimal design points (Figure 14). According to the ranking of optimal solutions on the obtained Pareto front, optimal solution 53, with the highest relative distance

Table 8. Number of function evaluations and run time needed in the proposed approach to solve the side-impact problem.

Number of function evaluations			
Dimension reduction method	Reliability evaluation	Total	Run time (s)
300 × 200 × 15 = 900,000	790,920,000	791,920,000	3.712356E4

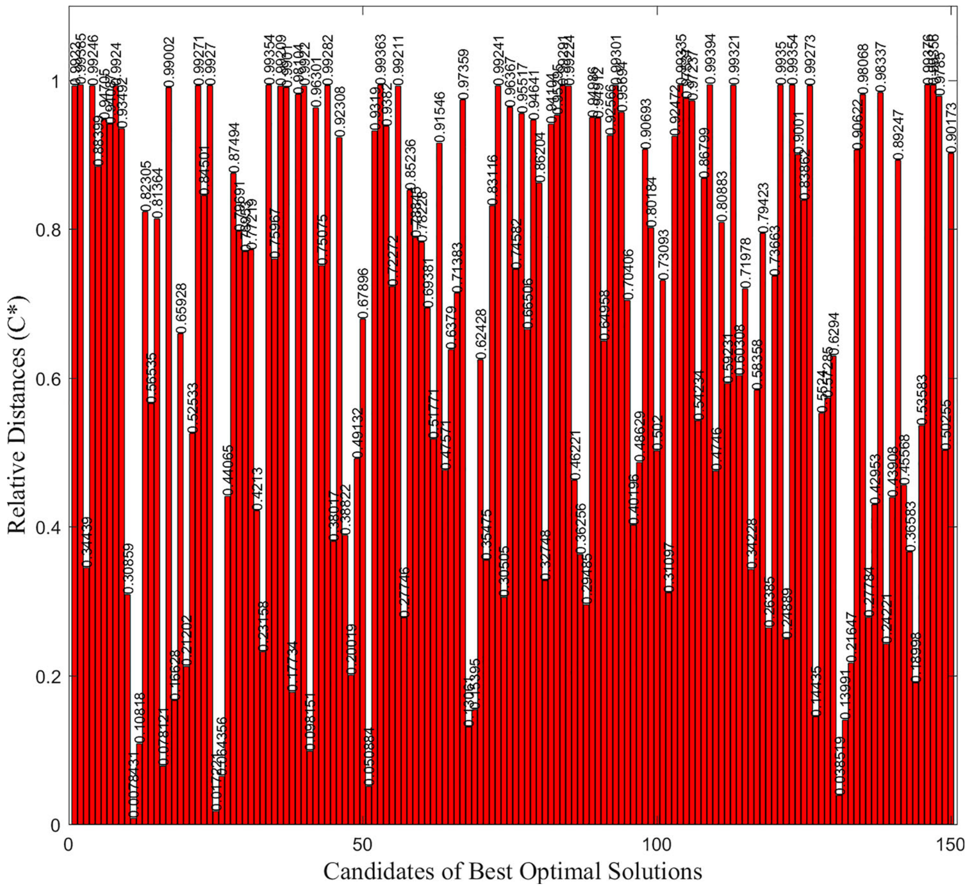

Figure 14. Ranking the optimal design points on the three-dimensional Pareto front (car side-impact test problem).

value ($C^* = 0.994$) is the closest optimal design point to the ideal solution (reliability = 0.9, cost function mean = 22 kg, variance = 0.12 kg^2).

To evaluate the proposed BRBRDO method, comparisons of the results obtained by the proposed method and other three approaches, *i.e.* DO, RDO (Arora 2004) and BRBDO (Srivastava and Deb 2013), are reported in Table 9. For this case study, the parameters of NSGA-II are adjusted as follows: population size = 200 individuals, generation number = 300 generations, crossover rate = 0.9 and mutation rate = 0.05.

Referring to Table 9, the DO approach leads to an unreliable design (reliability = 0.49). The RDO approach, which minimizes simultaneously the cost function mean and the robustness parameter, ignores feasibility insurance in the design process. Therefore, although RDO has attained a solution with a lower value of the robustness parameter compared with the proposed approach, it has failed to obtain a reliable solution (reliability = 0.48). In contrast, the proposed approach ensures reliability by making trade-offs between the cost function and the robustness parameter. According to Table 9,

Table 9. Comparison of the obtained optimum design using the proposed method and other approaches in the side-impact problem.

Approach	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Obj. 1: weight	Obj. 2: variance	Obj. 3: reliability
DO	0.500	1.226	0.450	1.228	0.875	0.896	0.400	23.32	0.130	0.49
RDO	1.025	1.413	0.599	1.405	1.617	0.782	0.701	31.04	0.086	0.48
BRBDO	0.524	1.415	0.457	1.285	1.044	1.162	0.407	24.94	0.170	0.81
Proposed algorithm (BRBRDO)	0.565	1.377	0.488	1.297	1.02	1.166	0.475	25.65	0.130	0.81

Note: DO = deterministic optimization; RDO = robust design optimization; BRBDO = Bayesian reliability-based design optimization; BRBRDO = Bayesian reliability-based robust design optimization.

Table 10. Relative distance to **IS*** of the proposed approach and Bayesian reliability-based design optimization (BRBDO) solutions in the helical spring problem.

Approach	R_D
Proposed approach	0.5213
BRBDO	0.9792

the proposed approach presents a solution with a higher reliability value, of 68.75%, and a lower cost function mean, by about 17.36%, compared to the RDO approach.

The results demonstrate that DO and RDO do not ensure the feasibility of the design in the presence of uncertainty. Therefore, only the BRBDO seems to be comparable with the proposed BRBRDO approach in handling the uncertainty. According to the obtained results, the variance of the solution obtained by the proposed method has been improved by 23.53%, the cost function mean value has been increased by about 2.85% and the reliability has not changed compared with the BRBDO method. To demonstrate the superiority of the solution obtained by the proposed approach over the BRBDO solution, the relative distances (Equation 30) of the obtained objectives from the ideal solution ($IS^* = 23.32, 0.086, 0.81$) are compared and reported in Table 10. The optimal objectives obtained by the proposed approach are closer to the ideal solution (IS^*) compared to the BRBDO optimal objectives.

4. Conclusions

In the literature, RBRDO approaches have been developed based on the assumption that all uncertainties are aleatory. Therefore, these methods are inefficient in dealing with epistemic uncertainty. In this article, a novel Bayesian RBRDO (BRBRDO) framework was proposed to solve engineering design problems involving both aleatory and epistemic uncertainties. In the proposed approach, to design optimal, robust and reliable products effectively, the design problem under uncertainty is rewritten as an equivalent multi-objective optimization problem. Then, the NSGA-II algorithm is utilized to obtain the optimal Pareto front. To approximate the objective functions at each stage of the optimization procedure, the Bayesian reliability technique and a conservative form of univariate DRM are utilized. Finally, the enhanced Shannon's entropy-based TOPSIS method was applied to select the best design point. To illustrate the capability of the presented approach, two case studies were considered. The results of the proposed method were compared to those obtained by some conventional methods in the literature. The results show that the proposed BRBRDO leads to a less sensitive solution than the results obtained by the BRBDO method. Also, compared with the results obtained by the RDO and DO methods, the proposed method can lead to more reliable designs. In other words, the design points that were obtained using the proposed BRBRDO algorithm are the best solution considering the optimality, robustness and reliability.

Disclosure statement

No potential conflict of interest was reported by the authors.

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